

INNER DERIVATIONS OF EXCEPTIONAL LIE ALGEBRAS IN PRIME CHARACTERISTIC

PABLO ALBERCA BJERREGAARD, DOLORES MARTÍN BARQUERO,
AND CÁNDIDO MARTÍN GONZÁLEZ

ABSTRACT. It is well-known that every derivation of a semisimple Lie algebra \mathfrak{L} over an algebraically closed field F with characteristic zero is inner. The aim of this paper is to show what happens if the characteristic of F is prime with \mathfrak{L} an exceptional Lie algebra. We prove that if L is a Chevalley Lie algebra of type $\{\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8\}$ over a field of characteristic p then the derivations of L are inner except in the cases \mathfrak{e}_6 with $p = 3$ and \mathfrak{e}_7 with $p = 2$.

1. INTRODUCTION

This paper deals with the derivation algebra of the Chevalley Lie algebras of exceptional type $\{\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8\}$. As it is well-known Chevalley constructed Lie algebras over arbitrary fields F starting from any semisimple (finite-dimensional) Lie algebra over an algebraically closed field of characteristic zero. The key point was to realize that on such algebras one can find a suitable basis whose structure constants are integers. Then, by an scalar extension process he constructed those Lie algebras which nowadays are called Chevalley algebras.

By gathering results of Zassenhaus (1939), Seligman (1979), Springer and Steinberg (we will give full details in the next section) one can get convinced that any derivation of a Chevalley F -algebra of any of the types $\{\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7\}$ is inner for $\text{char}(F) \geq 5$. Also the derivations of a Chevalley algebra of type \mathfrak{e}_8 with $\text{char}(F) \geq 7$, are inner.

In [1], the case \mathfrak{f}_4 over fields of characteristic $\neq 2$ is considered. There, it is proved that all derivations are inner. In this paper we study the cases not covered previously: the derivations of the algebras of type $\{\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7\}$ over fields of characteristic 2 or 3, and algebras of type \mathfrak{e}_8 over fields of characteristic 2, 3 or 5 (we overlap with [1] only in the case \mathfrak{f}_4 for characteristic 3 though our methodology in this case is different).

Returning to the reference [1], the key idea in this paper is the formula stating that any derivation is the sum of an inner derivation plus a derivation annihilating a fixed Cartan subalgebra ([1, Proof of Proposition 8.1]). We have proved that a certain extension of this idea is also true for Chevalley algebras of exceptional type: any derivation is the sum of an inner derivation plus a derivation mapping a fixed Cartan subalgebra into the center. Thus, when the center is null, we recover the formula in [1]. This formula reduces the computation of the dimension of the derivation algebra to the computation of the dimension of the vector space V of derivations mapping a fixed Cartan subalgebra into the center.

In any case we have proved that V has a very low dimension, but in spite of this, the computations needed to determine $\dim V$ have been specially complex in the case of \mathfrak{e}_8 .

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For the exceptional algebras of low rank \mathfrak{g}_2 and \mathfrak{f}_4 , most of the computational algebra packages with support for Lie algebras gives us directly the dimension of the derivation algebra. This is the case of GAP which has been our choice for these computations. However this brute-force approach does not work for the algebras \mathfrak{e}_i , $i = 6, 7, 8$ which have a higher rank. In this cases even the powerful computing servers that we tried run out of memory almost immediately. Thus, a different strategy was needed and the solution came of the hand of the space V which complements the ideal of inner derivations. So our approach is based on: (1) reducing the computation of the dimension of the derivation algebra to the computation of the dimension of V ; (2) computing $\dim(V)$ (which is the difficult part).

For the computation of $\dim(V)$ we needed to let the main routine to run under *Mathematica* of Wolfram Reseach, for 5 minutes with \mathfrak{e}_6 , an hour with \mathfrak{e}_7 , and almost 3 days with \mathfrak{e}_8 .

2. PREVIOUS DEFINITIONS AND RESULTS

Let k be an algebraically closed field k of characteristic 0. Recall that the Cartan matrix of a finite-dimensional semi-simple Lie algebra \mathfrak{L} over k is a matrix

$$A = \left(2 \frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \right)_{i,j=1,\dots,r}$$

where $\alpha_1, \dots, \alpha_r$ is some system of simple roots of \mathfrak{L} with respect to a fixed Cartan subalgebra \mathfrak{t} and $(\ , \)$ is the scalar product on the dual space of \mathfrak{t} defined by the Killing form on \mathfrak{L} . The entries $a_{ij} = 2 \frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)}$ of a Cartan matrix have the following properties:

$$(1) \quad \left. \begin{array}{l} a_{ii} = 2; \quad a_{ij} \leq 0 \text{ and } a_{ij} \in \mathbb{Z} \text{ for } i \neq j, \\ a_{ij} = 0 \Rightarrow a_{ji} = 0. \end{array} \right\}$$

The Cartan matrix is a key tool in the description of \mathfrak{L} by generators and relations. Denote by Φ the root system of \mathfrak{L} relative to K . As it is known, there exist in \mathfrak{L} a basis $\{v_\alpha\}_{\alpha \in \Phi} \cup \{h_i\}_{i=1}^r$ whose structure constants are in \mathbb{Z} :

- (1) $[h_i, h_j] = 0$, for any $i, j \in \{1, \dots, r\}$.
- (2) $[h_i, v_\alpha] = \langle \alpha, \alpha_i \rangle v_\alpha \in \mathbb{Z} v_\alpha$ for $\alpha \in \Phi$, $1 \leq i \leq r$ and $\langle \alpha, \beta \rangle := 2 \frac{(\alpha, \beta)}{(\beta, \beta)}$.
- (3) $[v_\alpha, v_{-\alpha}] = h_\alpha$ a \mathbb{Z} -linear combination of h_1, \dots, h_r .
- (4) If α and β are independent roots, $\beta - r\alpha, \dots, \beta + q\alpha$ the α -string through β then $[v_\alpha, v_\beta] = 0$ if $q = 0$ while $[v_\alpha, v_\beta] = \pm(r+1)v_{\alpha+\beta}$ if $\alpha + \beta \in \Phi$.

Such a basis $\{v_\alpha\}_{\alpha \in \Phi} \cup \{h_i\}_{i=1}^r$ is called a *Chevalley basis* of \mathfrak{L} and its existence is proved for instance in [6, Proposition, p. 146 and Theorem, p. 147].

Following [6, p. 149], given a Chevalley basis $\{v_\alpha\}_{\alpha \in \Phi} \cup \{h_i\}_{i=1}^r$ of the semisimple Lie algebra \mathfrak{L} over k we can consider its linear envelope $\mathfrak{L}_{\mathbb{Z}}$ which is a Lie algebra over \mathbb{Z} in the obvious sense. Now if \mathbf{F}_p is the field of integers module the prime p we can construct the Lie \mathbf{F}_p -algebra $\mathfrak{L}_{\mathbf{F}_p} := \mathfrak{L}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbf{F}_p$. The multiplication table of this algebra in a suitable basis is the one for \mathfrak{L} reduced module p . Now, if F is any extension field of \mathbf{F}_p we can construct the Lie algebra (over F) given by $\mathfrak{L}_F := \mathfrak{L}_{\mathbf{F}_p} \otimes_{\mathbf{F}_p} F$. This algebra inherits both basis and Lie algebra structure from $\mathfrak{L}_{\mathbf{F}_p}$. In this way we obtain from each couple (\mathfrak{L}, F) a Lie algebra \mathfrak{L}_F over F , called a *Chevalley algebra*, which depends up to isomorphism only on \mathfrak{L} and the field F . If \mathfrak{L} is simple of Dynkin diagram D , then we will say that \mathfrak{L}_F is a Lie algebra of type D . A common notation of this algebra will be $D(F)$ (for instance $\mathfrak{f}_4(\mathbf{F}_3)$ or $\mathfrak{e}_8(\mathbf{F}_2)$).

Let \mathfrak{L} be a Lie algebra over a field F . If $x \in \mathfrak{L}$, we have $\text{ad}(x) : \mathfrak{L} \rightarrow \mathfrak{L}$ with $\text{ad}(x)(y) := [x, y]$. It is easy to check that $\text{ad}(x) \in \text{Der}(\mathfrak{L})$. We will denote $\text{ad}(\mathfrak{L}) := \{\text{ad}(x) : x \in \mathfrak{L}\} \subset \text{Der}(\mathfrak{L})$. In fact,

$$(1) \quad \text{ad}(\mathfrak{L}) \triangleleft \text{Der}(\mathfrak{L}).$$

Theorem 1. *Let F be an algebraically closed field of characteristic zero. If \mathfrak{L} is a finite-dimensional semisimple Lie algebra over F , then*

$$(2) \quad \text{ad}(\mathfrak{L}) = \text{Der}(\mathfrak{L})$$

If \mathfrak{L} is finite-dimensional, we have the *Killing form* of \mathfrak{L} :

$$(3) \quad k : \mathfrak{L} \times \mathfrak{L} \rightarrow F, \quad k(x, y) := \text{tr}(\text{ad}(x), \text{ad}(y)),$$

and we have the following results

Theorem 2 (Zassenhaus [8]). *If \mathfrak{L} is a finite-dimensional Lie algebra with non-degenerate Killing form, then every derivation is inner. That is to say*

$$(4) \quad \text{ad}(\mathfrak{L}) = \text{Der}(\mathfrak{L}).$$

Let \mathfrak{L} be a Chevalley Lie algebra of exceptional type $\{\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8\}$. A very fundamental result ([5, p.49]) in what follows, due to Seligman and independently to Springer and Steinberg, says that (relative to a Chevalley basis)

$$(5) \quad \det k = 2^\alpha 3^\beta, \quad \text{if } \mathfrak{L} \text{ is of type } \mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6 \text{ or } \mathfrak{e}_7,$$

$$(6) \quad \det k = 2^\alpha 3^\beta 5^\gamma, \quad \text{if } \mathfrak{L} \text{ is of type } \mathfrak{e}_8,$$

for some nonzero scalars α, β and γ . It is easy now to obtain

Corollary 1. *Let \mathfrak{L} be a Chevalley F -algebra of any of the types $\{\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7\}$, then if $\text{char}(F) \geq 5$ every derivation is inner. If \mathfrak{L} is of type \mathfrak{e}_8 and $\text{char}(F) \geq 7$, every derivation in \mathfrak{L} is inner.*

We have defined $\text{ad} : \mathfrak{L} \rightarrow \text{Der}(\mathfrak{L})$ as $x \mapsto \text{ad}(x)$. An important fact is that

$$(7) \quad \dim \text{ad}(\mathfrak{L}) = \dim \mathfrak{L} - \dim \ker \text{ad} = \dim \mathfrak{L} - \dim Z(\mathfrak{L}),$$

where $Z(\mathfrak{L})$ is the *center* of the Lie algebra. If, for example, $Z(\mathfrak{L}) = 0$ then ad is a monomorphism and $\dim \mathfrak{L} = \dim \text{ad}(\mathfrak{L}) \leq \dim \text{Der}(\mathfrak{L})$. Now if $\dim \mathfrak{L} = \dim \text{Der}(\mathfrak{L})$ then $\text{ad}(\mathfrak{L}) = \text{Der}(\mathfrak{L})$, and every derivation will be inner.

Take now a simple Lie algebra \mathfrak{L} over an algebraically closed field k of characteristic zero. Fix a Chevalley basis $\{v_\alpha\} \cup \{h_i\}$ of \mathfrak{L} . The different h_i generate a Cartan subalgebra H and each v_α the root space \mathfrak{L}_α . Consider now an arbitrary field F and consider the Chevalley algebra \mathfrak{L}_F introduced above. The properties of the new algebra \mathfrak{L}_F may have changed dramatically, for instance \mathfrak{L}_F may happen to be non-simple. Also while \mathfrak{L} is simple, the algebra \mathfrak{L}_F may have a nonzero center. In any case $\mathfrak{L}_F = H_F \oplus (\oplus_\alpha (\mathfrak{L}_\alpha)_F)$ where H_F is the F -linear span of the h_i 's and $(\mathfrak{L}_\alpha)_F$ is the one-dimensional space Fv_α .

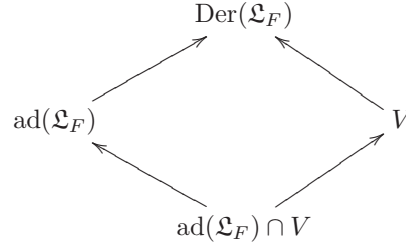
Remark 1. If $\alpha : H \rightarrow k$ is a nonzero root of \mathfrak{L} then we can define a root $\bar{\alpha}$ of \mathfrak{L}_F taking into account that H_F is generated as F -vector space by the h_i 's of the Chevalley basis $\{v_\alpha\}_{\alpha \in \Phi} \cup \{h_i\}_{i=1}^r$. We must realize that $\alpha(h_i) \in \mathbb{Z}$ by (2) of the definition of Chevalley basis. Thus, we define $\bar{\alpha} : H_F \rightarrow F$ as the F -linear extension $\bar{\alpha}(\sum_i \lambda_i h_i) := \sum_i \lambda_i \alpha(h_i)$. Since α is nonzero $\bar{\alpha}$ is also a nonzero. Furthermore it is straightforward that for any $h \in H_F$ one has $[h, v_\alpha] = \bar{\alpha}(h)v_\alpha$ in \mathfrak{L}_F .

The result in the remark below is a corollary of the main result in [2].

Remark 2. Let \mathfrak{L}_F be a Chevalley algebra of exceptional type over a field F of prime characteristic. Then the center of \mathfrak{L}_F is zero except for \mathfrak{e}_6 in characteristic 3 and \mathfrak{e}_7 in characteristic 2. In this cases the center Z is one-dimensional and $Z \subset H_F$.

The following results takes the idea of [1, Proposition 8.1] to whom we are emdebted. Though in [1] the result appears in a particular context, the idea can be extended to the following setting.

Theorem 3. *Let \mathfrak{L} be a semisimple Lie algebra over an algebraically closed field k of characteristic zero and let \mathfrak{L}_F be the corresponding Chevalley F -algebra (where F is a field of prime characteristic). Denote by V the F -vector space of all derivations d of \mathfrak{L}_F such that $d(H_F) \subset Z := Z(\mathfrak{L}_F)$, then $\text{Der}(\mathfrak{L}_F) = \text{ad}(\mathfrak{L}_F) + V$ hence we have the following Hesse diagram of subspaces*



Proof. Consider first the Cartan decomposition $\mathfrak{L} = H \oplus (\oplus_{\alpha} \mathfrak{L}_{\alpha})$ of \mathfrak{L} with relation to H . As it is known this decomposition is a (fine) grading on \mathfrak{L} whose zero component is H and the rest of the homogeneous components are the root spaces \mathfrak{L}_{α} . Moreover, it is a group grading with grading group \mathbb{Z}^r being $r = \dim H$. Next we take the induced \mathbb{Z}^r -grading on the Chevalley algebra (zero component H_F and homogeneous components $(\mathfrak{L}_{\alpha})_F$) which will play an important role in the forthcoming argument. Now we consider the \mathbb{Z}^r -grading on $\text{Der}(\mathfrak{L}_F)$ such that a derivation d is of degree z if and only if d takes the component of degree w of \mathfrak{L}_F to the component of degree $w + z$ (for any $z, w \in \mathbb{Z}^r$). In order to prove the formula $\text{Der}(\mathfrak{L}_F) = \text{ad}(\mathfrak{L}_F) + V$ it suffices to prove that for any homogeneous derivation d we can decompose d as $d = \text{ad}(x) + v$ for some $x \in \mathfrak{L}_F$ and $v \in V$. We analyze first the case in which d is of degree 0. Then $d(H_F) \subset H_F$ and $d(v_{\alpha}) = \lambda_{\alpha} v_{\alpha}$ for some scalar $\lambda_{\alpha} \in F$. So starting from $[h, v_{\alpha}] = \alpha(h)v_{\alpha}$ and applying d we get $[d(h), v_{\alpha}] + \lambda_{\alpha}[h, v_{\alpha}] = \alpha(h)\lambda_{\alpha}v_{\alpha}$ (for arbitrary α). Thus $[d(h), v_{\alpha}] = 0$ and $d(h) \in Z(\mathfrak{L}_F)$ so that in this case $d \in V$. Now assume that d is a derivation of degree $z \neq 0$ and that $d(H_F) \neq 0$ (if d annihilates H_F then $d \in V$). If the component of degree z of \mathfrak{L}_F is $(\mathfrak{L}_{\alpha})_F$ then for any $h \in H_F$ we have $d(h) = \lambda(h)v_{\alpha}$ where $\lambda: H_F \rightarrow F$ is a linear map. If we now take arbitrary elements $h, k \in H_F$, since $[h, k] = 0$ we get

$$\lambda(h)[v_{\alpha}, k] + \lambda(k)[h, v_{\alpha}] = 0,$$

implying $\lambda(h)\bar{\alpha}(k) = \lambda(k)\bar{\alpha}(h)$. The fact that d does not annihilates H_F implies $\lambda \neq 0$ hence there is some $h \in H_F$ such that $\bar{\alpha} = c\lambda$ being $c = \lambda(h)^{-1}\bar{\alpha}(h)$. Take into account also that $c \neq 0$ since $\bar{\alpha} \neq 0$ as pointed out in Remark 1. Finally the reader can check that $d + c^{-1}\text{ad}(v_{\alpha})$ is a derivation of \mathfrak{L}_F in V (more precisely it annihilates H_F).

Corollary 2. *If Z is the center of \mathfrak{L}_F then*

$$\dim \text{Der}(\mathfrak{L}_F) = \dim_k \mathfrak{L} + \dim V - \dim_k H.$$

(we write \dim for \dim_F).

Proof. We know that $\dim \text{ad}(\mathfrak{L}_F) = \dim \mathfrak{L}_F - \dim Z = \dim_k \mathfrak{L} - \dim Z$. On the other hand $\text{ad}(\mathfrak{L}_F) \cap V = \{ad(x): [x, H_F] \subset Z\}$ therefore $\dim \text{ad}(\mathfrak{L}_F) \cap V = \dim\{x: [x, H_F] \subset Z\} - \dim Z$. Next we prove that $\{x: [x, H_F] \subset Z\} = H$. Take $x = h + \sum_{\alpha} \lambda_{\alpha} v_{\alpha}$ with $h \in H_F$ and $\lambda_{\alpha} \in F$, satisfying $[x, H_F] \subset Z$. Then $\sum_{\alpha} \lambda_{\alpha} \alpha(k) v_{\alpha} \in Z$ for any $k \in H$. Since $Z \subset H_F$ we conclude $\sum_{\alpha} \lambda_{\alpha} \alpha(k) v_{\alpha} = 0$ hence if some $\lambda_{\alpha} \neq 0$ then $\alpha(H) = 0$ and so $\alpha = 0$ a contradiction.

So $\dim \text{Der}(\mathfrak{L}_F) = \dim_k \mathfrak{L} - \dim Z + \dim V - \dim_k H + \dim Z$ whence the result.

We finish this section describing a general algorithm for computing a basis of any finite-dimensional algebra A over a field K , starting from a linearly independent set S of algebra generators. We recall that a subset $S \subset A$ is a generator system for A if any element in A is a linear combination of products of elements in S . Denote by \mathfrak{G} the set of all finite generator systems of A which are linearly independent. Since A is finite-dimensional \mathfrak{G} is nonempty. Define $H: \mathfrak{G} \rightarrow \mathfrak{G}$ the map $S \mapsto H(S)$ where $H(S)$ is computed in the following way: Take $S = \{x_1, \dots, x_n\}$ and construct the matrix M whose (i, j) -entry is the product $x_i x_j$. Write now the matrix M as a vector of n^2 coordinates $M = (l_1, \dots, l_{n^2})$. If $l_1, \dots, l_{q-1} \in \text{span}_K(S)$ and $l_q \notin \text{span}_K(S)$ then define $H(S) := S \cup \{l_q\}$. In other words $H(S)$ is obtained adjoining to S the first element in M which is not in the linear span of S .

By construction $H(S)$ is linearly independent (hence finite) and it is a system of generators since it contains S . The following result will be crucial for our purposes:

Proposition 1. *For any $S \in \mathfrak{G}$ there is a unique fixed point B for H such that $S \subset B$. Moreover B is a basis of the algebra A .*

Proof. The sequence $S, H(S), H^2(S), \dots$ stabilizes necessarily because of the finite-dimensionality of A . Thus if $H^{n+1}(S) = H^n(S)$ then $B := H^n(S)$ is a fixed point of H and $S \subset B$. By construction B is linearly independent. To prove that B is a basis of A take into account that by the definition of H , for any $x, y \in S$ we have $xy \in \text{span}_K(H(S))$. In particular the product of elements of S is in the linear span $\text{span}_K(B)$. Since any element in the algebra is a linear combination of products of elements in S we conclude that $\text{span}_K(B) = A$. Let us prove now that B is the unique fixed point of H containing S . Assume that $B' \in \mathfrak{G}$ satisfies $S \subset B'$ and $H(B') = B'$. Then $H(S) \subset B'$ and applying H once and again $H^n(S) \subset B'$ for any n . Thus $B \subset B'$. Since B' is linearly independent and B a basis we conclude $B = B'$.

3. EXCEPTIONAL LIE ALGEBRAS \mathfrak{g}_2 AND \mathfrak{f}_4

With these two algebras, \mathfrak{g}_2 and \mathfrak{f}_4 , we have used the software GAP to compute the mentioned dimensions. All the following routines have run reasonably rapidly in a personal computer.

3.1. Lie algebra \mathfrak{g}_2 . Let us consider a Chevalley Lie algebra \mathfrak{g}_2 . If the characteristic of the base field is other than 2 or 3 then every derivation is inner (Corollary 1). We present a computational approach to the problem and extend this result to the cases not covered. We are going to prove the following result:

Theorem 4. *Let $\mathfrak{L} = \mathfrak{g}_2$ over a field F of prime characteristic p . Every derivation in \mathfrak{g}_2 is inner if, and only if, $p \neq 2$. Thus*

$$(8) \quad \text{ad}(\mathfrak{g}_2) = \text{Der}(\mathfrak{g}_2) \text{ if } \text{char}(F) \neq 2.$$

If the characteristic is 2 then $\dim \text{Der}(\mathfrak{g}_2) = 21$ while $\dim \mathfrak{g}_2 = 14$.

PROOF. If the characteristic of the base field is $p \geq 5$ we can use Corollary 1 and (5), and then every derivation is inner. Let us now consider $p = 3$, since

$\dim \text{Der}(\mathfrak{g}_2(F)) = \dim \text{Der}(\mathfrak{g}_2(\mathbf{F}_3))$, we may take without loss of generality $F = \mathbf{F}_3$. We use the software GAP. The next lines

```
gap> F:=GF(3);
gap> L:=SimpleLieAlgebra("G",2,F);
<Lie algebra of dimension 14 over GF(3)>
```

define the base field F of characteristic 3 and the Lie algebra \mathfrak{g}_2 over F , which is 14-dimensional. We can also compute the center of the Lie algebra by doing

```
gap> LieCenter(L);
<Lie algebra of dimension 0 over GF(3)>
```

which is 0-dimensional. We determine the dimension of the Lie algebra $\text{Der}(\mathfrak{g}_2)$:

```
gap> B:=Basis(L);
CanonicalBasis( <Lie algebra of dimension 14 over GF(3)> )
gap> Derivations(B);
<Lie algebra of dimension 14 over GF(3)>
```

and then $\dim \text{Der}(\mathfrak{g}_2) = 14$. We have $\dim \text{ad}(\mathfrak{g}_2) = \dim \mathfrak{g}_2 - \dim Z(\mathfrak{g}_2) = 14 - 0 = 14 = \dim \text{Der}(\mathfrak{g}_2)$. Thus every derivation is inner also if $\text{char}(F) = 3$ as we have confirmed that

$$(9) \quad \text{ad}(\mathfrak{g}_2) = \text{Der}(\mathfrak{g}_2), \text{char}(F) = 3,$$

in this case by using a dimensional reasoning.

Now, if $p = 2$, we do first

```
gap> F:=GF(2);
gap> L:=SimpleLieAlgebra("G",2,F);
<Lie algebra of dimension 14 over GF(2)>
```

in order to work with \mathfrak{g}_2 with $\text{char}(F) = 2$, which is also 14-dimensional. We have now to define a basis and then we can compute the dimension of $\text{Der}(\mathfrak{g}_2)$:

```
gap> B:=Basis(L);
CanonicalBasis( <Lie algebra of dimension 14 over GF(2)> )
gap> Derivations(B);
<Lie algebra of dimension 21 over GF(2)>
```

and it has dimension 21. Then

$$(10) \quad \text{ad}(\mathfrak{g}_2) \subsetneq \text{Der}(\mathfrak{g}_2), \text{char}(F) = 2,$$

because $\dim \text{ad}(\mathfrak{g}_2) = \dim \mathfrak{g}_2 = 14 \neq 21$, where we have used that the center is again 0-dimensional, which can be confirmed with the next code line:

```
gap> LieCenter(L);
<Lie algebra of dimension 0 over GF(2)>
```

Finally, we present a table that summarizes the previous computations:

\mathfrak{L}	char.	$\dim \mathfrak{L}$	$\dim Z(\mathfrak{L})$	$\dim \text{ad}(\mathfrak{L})$	$\dim \text{Der}(\mathfrak{L})$	$\text{ad}(\mathfrak{L}) = \text{Der}(\mathfrak{L})$
\mathfrak{g}_2	2	14	0	14	21	no
\mathfrak{g}_2	3	14	0	14	14	yes

□

3.2. Lie algebra \mathfrak{f}_4 . Let us work with the Chevalley Lie algebra \mathfrak{f}_4 . If the characteristic of the base field is other than 2 or 3, we can use Corollary 1 to conclude that every derivation is inner. In spite of the fact that A. Elduque and M. Kochetov have recently proved that the result is also true if the characteristic of the base field is 3 (see [1, Proposition 8.1]) we present an extension that confirms this result is also true at any prime characteristic.

If the characteristic of the base field is 2 or 3 we have repeated the previous strategy using GAP with the results summarized in the following table:

\mathfrak{L}	char.	$\dim \mathfrak{L}$	$\dim Z(\mathfrak{L})$	$\dim \text{ad}(\mathfrak{L})$	$\dim \text{Der}(\mathfrak{L})$	$\text{ad}(\mathfrak{L}) = \text{Der}(\mathfrak{L})$
\mathfrak{f}_4	2	52	0	52	52	yes
\mathfrak{f}_4	3	52	0	52	52	yes

Then we have the next result:

Theorem 5. *If $\mathfrak{L} = \mathfrak{f}_4$ over a field F of prime characteristic then*

$$(11) \quad \text{ad}(\mathfrak{f}_4) = \text{Der}(\mathfrak{f}_4),$$

that is to say, every derivation of \mathfrak{f}_4 is inner.

4. EXCEPTIONAL LIE ALGEBRAS \mathfrak{e}_6 , \mathfrak{e}_7 AND \mathfrak{e}_8

In these cases we needed to implement certain algorithms with the software *Mathematica* as we ran out of memory using GAP (even with a powerful machine), overcoat from \mathfrak{e}_7 but specially with \mathfrak{e}_8 .

As we have increased the dimensionality of the Lie algebras, 78, 133 and 248, we had to decide to use a powerful computer system, called *Picasso* and it is a support given by the University of Malaga (see the acknowledgments) or a new strategy. We soon ran out of memory working with the old strategy, direct calculations, and we had to use a new simplification of the problem.

The main idea now, following Section 2 and Theorem 3, is to use the decomposition $\text{Der}(\mathfrak{L}_F) = \text{ad}(\mathfrak{L}_F) + V$ given in Theorem 3. Finally, with all the ingredients presented, the strategy seems to be clear. We will use the formula in Corollary 2 to compute the dimension of $\text{Der}(\mathfrak{L})$ and compare it with the dimension of $\text{ad}(\mathfrak{L})$, given by (7). If these two quantities agree, we have that every derivation is inner.

That is why we will use a Cartan decomposition of \mathfrak{L}_F . We need a Cartan subalgebra H_F and the center Z_F , their dimensions, and the dimension of V , the space of all derivations d with $d(H_F) \subset Z_F$. These derivations leave the root spaces d -invariant.

First of all we need to compute a basis of the Lie algebra. This is done from a system of generators obtained by [4] in the \mathfrak{e}_6 and \mathfrak{e}_7 cases, and from GAP in the \mathfrak{e}_8 case. We can find then a basis and then a new one from the Cartan decomposition, where the matrix of $d \in V$ is block-diagonal. The center $Z(\mathfrak{L}_F)$ and $\text{ad}(\mathfrak{L}_F) \cap V$ can be also finally determined.

We have used *Mathematica* with all the following computations.

4.1. Lie algebra \mathfrak{e}_6 . As shown in (5) we have that the killing form has determinant $2^\alpha 3^\beta$, for some $\alpha, \beta \in F$. If $\text{char}(F) \neq 2, 3$, every derivation is inner by Theorem 2.

We have a list S of generators of the Lie algebra from [4, §3.2]:

$$(12) \quad S = \{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_1^t, \phi_2^t, \phi_3^t, \phi_4^t, \phi_5^t, \phi_6^t\},$$

where

$$\begin{aligned}
\phi_1 &= E_{1,2} + E_{11,13} + E_{14,16} + E_{17,18} + E_{19,20} + E_{21,22}, \\
\phi_2 &= E_{4,5} + E_{6,7} + E_{8,10} + E_{19,21} + E_{20,22} + E_{23,24}, \\
\phi_3 &= E_{2,3} + E_{9,11} + E_{12,14} + E_{15,17} + E_{20,23} + E_{22,24}, \\
\phi_4 &= E_{3,4} + E_{7,9} + E_{10,12} + E_{17,19} + E_{18,20} + E_{24,25}, \\
\phi_5 &= E_{4,6} + E_{5,7} + E_{12,15} + E_{14,17} + E_{16,18} + E_{25,26}, \\
\phi_6 &= E_{6,8} + E_{7,10} + E_{9,12} + E_{11,14} + E_{13,16} + E_{26,27},
\end{aligned}$$

with $E_{i,j}$ the elementary matrices of order 27 and the superscript $(\cdot)^t$ denotes matrix transposition.

Let us consider now the case $\text{char}(F) = 2$. We obtain then a basis applying Proposition 1 to S . We construct this way a basis $B = \{b_i\}_{i=1}^{78}$ of \mathfrak{e}_6 . This basis is available in the link <http://www.matap.uma.es/~alberca/Be6char2.txt>.

Now it is time to search for a Cartan subalgebra. We find first that the set

$$(13) \quad H = \{h_i\}_{i=1}^6 = \{b_{14}, b_{19}, b_{21}, b_{22}, b_{23}, b_{26}\}$$

is abelian and, as its cardinal is 6 (the dimension of any Cartan subalgebra of \mathfrak{e}_6), we only have to prove that all the maps $\text{ad}h_i$, $i = 1, \dots, 6$ are diagonalizable. We define first some auxiliary commands:

```

c[x_, y_] := x.y - y.x;
char = 2;
ξ[x_] := Module[v, v = Table[λi, {i, 78}];
v //. ToRules[Reduce[x == Sum[λiB[[i]], {i, 78}], Modulus → char]]];
elem[x_] := Sum[x[[i]]B[[i]], {i, 78}];
Rec[x_] := ξ[x].Table["b"i, {i, 78}];

```

the Lie bracket, coordinates, the element from its coordinates and the last one to write an element of the Lie algebra as a linear combination of the b_i , $i = 1, \dots, 78$.

Now we confirm that H is a Cartan subalgebra as we have

$$(14) \quad \text{ad}h_i(b_j) = \lambda_{i,j}b_j,$$

and the $\text{ad}h_i$ are diagonalizable for all $i = 1, \dots, 6$. We compute the center of the Lie algebra. The code is as simply as:

```

gcL = Sum[λiB[[i]], {i, 78}];
Do[
gcL = (Expand[
gcL //. ToRules[
Eliminate[
Union[Flatten[c[gcL, B[[i]]]] == 0 && Modulus == md, ]],
Modulus → md]), {i, 78}];
Length[Variables[gcL]]

```

and we get no variables left and then $\dim Z(\mathfrak{e}_6) = 0$. Now with the code

```

Do[
ES[i] = {};
Do[
a = c[Cartan[[i]], B[[j]]]; AppendTo[ES[i], {{i, j}, Rec[a]}, {j, 78}], {i, 6}]

```

we have the list $\{\{i, j\}, \lambda_{i,j}, b_j\}$. We can construct the root spaces \mathfrak{L}_α by collecting all the base elements that have the same scalars $\lambda_{i,j}$, for all h_i . The root spaces, denoted by V_i in the *Mathematica* code, are built by collecting the mentioned base elements b_i . The *Mathematica* code is

```

CU[l_, i_] := ReplacePart[l, Position[l, "b"i] → 1];
ele = {14, 19, 21, 22, 23, 26};
ind = Table[i, {i, 78}];
ind = Complement[ind, ele]; i = 1;
While[ind! = {},
  lista[i] = ind; j = ind[[1]]; ind = Complement[ind, {j}];
  Vi = {"b"j}; k = 1;
  αi = CU[Table[ES[l][[j, 2]], {1, 6}], j];
  While[k ≤ 78,
    If[k! = j && MemberQ[ind, k],
      If[CU[Table[ES[l][[k, 2]], {1, 6}], k] == αi,
        AppendTo[Vi, "b"k];
        ind = Complement[ind, {k}]; k ++, k ++]; i ++]

```

and we obtain 36 root spaces, all of them 2-dimensional. Thus, we have constructed this way a Cartan decomposition of \mathfrak{e}_6 if the characteristic of the base field is two:

$$(15) \quad \mathfrak{e}_6 = H \oplus \left(\bigoplus_{i=1}^{36} V_i \right), \quad |H| = 6, \quad |V_i| = 2, \quad i = 1, \dots, 36.$$

We define a new basis, BC , from this decomposition, and we can compute $\dim V$. As $Z(\mathfrak{e}_6) = 0$, the matrix of a derivation $d : \mathfrak{e}_6 \rightarrow \mathfrak{e}_6$ is a 78×78 block diagonal, in this new basis, as

$$(16) \quad M = \text{diag}(0_{6 \times 6}, X_1, \dots, X_{36}), \quad X_i = \begin{pmatrix} x_{1,i} & x_{2,i} \\ x_{3,1} & x_{4,i} \end{pmatrix},$$

because if $d \in V$, then $d(H) \subset Z = 0$ and the root spaces are d -invariant.

This matrix must verify the derivation condition. We only have to count the number of the remaining variables of M after solving

$$(17) \quad d([s_i, b'_j]) - [d[s_i], b'_j] - [s_i, d(b'_j)] = 0,$$

for $s_i \in S$, $i = 1, \dots, 6$ and $b'_j \in BC$, $j = 1, \dots, 6$. The routine is

```

md = 2;
Char2[mat_] := Module[{mm = mat},
  Do[
    If[IntegerQ[mm[[i, j]]], mm[[i, j]] = Mod[mm[[i, j]], 2],
    {i, Length[mm]}, {j, Length[mm]}]; mm];

M = Table[0, {i, 78}, {j, 78}];
Do[M[[i, i]] = x1,i; M[[i, i + 1]] = x2,i;
  M[[i + 1, i]] = x3,i; M[[i + 1, i + 1]] = x4,i, {i, 7, 77, 2}];
ξC[x_] := Module[{v}, v = Table[λi, {i, 78}];
  v //. ToRules[
    Reduce[Char2[x] == Sum[λi BC[[i]], {i, 78}], Modulus → md]]];
elemC[x_] := Sum[x[[i]] BC[[i]], {i, 78}];
d[x_] := elemC[ξC[x].M];
ec[i_, j_] :=
Expand[d[c[S[[i]], BC[[j]]]] - c[d[S[[i]], BC[[j]]] - c[S[[i]], d[BC[[j]]]], Modulus → md];

k = 1;
Do[
  M = (Expand[
    M //. ToRules[
      Eliminate[Union[Flatten[ec[i, j]] == 0 && Modulus == md, ]],
      Modulus → md]), {i, Length[S]}, {j, 78}];
Length[M]

```

and the result is $\dim V = 6$.

We have all the dimensions computed and we can affirm that

$$(18) \quad \begin{aligned} \dim \operatorname{Der}(\mathfrak{e}_6) &= \dim \mathfrak{e}_6 + \dim V - \dim H = \\ &= 78 + 6 - 6 = 78. \end{aligned}$$

Thus, as $\dim \operatorname{ad}(\mathfrak{e}_6) = \dim \mathfrak{e}_6 - \dim Z(\mathfrak{e}_6) = 78$, every derivation is inner in this case, with $\operatorname{char}(F) = 2$.

We have repeated all this computation if $\operatorname{char}(F) = 3$. We had to recalculate a basis from the generators (<http://www.matap.uma.es/~alberca/Be6char3.txt>) but after that we only had to change the characteristic of the base field and execute all the code. We had the following Cartan decomposition, if the characteristic is 3:

$$(19) \quad \mathfrak{e}_6 = H \oplus \left(\bigoplus_{i=1}^{72} V_i \right), \quad |H| = 6, \quad |V_i| = 1, \quad i = 1, \dots, 72,$$

with 1-dimensional root spaces and

$$(20) \quad H = \{b_{15}, b_{17}, b_{18}, b_{21}, b_{24}, b_{25}\},$$

a Cartan subalgebra. The matrix of a derivation in V does not have now a zero 6×6 block, because the center is

$$(21) \quad Z(\mathfrak{e}_6) = \langle 2b_{15} + b_{17} + 2b_{18} + b_{25} \rangle, \quad \operatorname{char}(F) = 3.$$

Thus, as if $d \in V$ we must have that $d(H) \subset Z$, each i -row of this previously null 6×6 -block is now as $(2\beta_i \quad \beta_i \quad 2\beta_i \quad 0 \quad 0 \quad \beta_i)$. We obtained again that $\dim V = 6$, and $\dim \operatorname{Der}(\mathfrak{e}_6) = 78$, but, as the dimension of the center is 1, we have $\dim \operatorname{ad}(\mathfrak{e}_6) = \dim \mathfrak{e}_6 - \dim Z(\mathfrak{e}_6) = 78 - 1 = 77 \neq \dim \operatorname{Der}(\mathfrak{e}_6)$.

We have then the following result:

\mathfrak{L}	char.	$\dim \mathfrak{L}$	$\dim Z(\mathfrak{L})$	$\dim \operatorname{ad}(\mathfrak{L})$	$\dim \operatorname{Der}(\mathfrak{L})$	$\operatorname{ad}(\mathfrak{L}) = \operatorname{Der}(\mathfrak{L})$
\mathfrak{e}_6	2	78	0	78	78	yes
\mathfrak{e}_6	3	78	1	77	78	no

and

Theorem 6. *Let $\mathfrak{L} = \mathfrak{e}_6$ over a field F of prime characteristic p . Every derivation of \mathfrak{e}_6 is inner, if and only if, $p \neq 3$. For $p = 3$ we have $\dim \operatorname{Der}(\mathfrak{e}_6) = 78$ and $\dim \operatorname{ad}(\mathfrak{e}_6) = 77$.*

4.2. Lie algebra \mathfrak{e}_7 . If the characteristic of the base field is not 2 or 3, every derivation is inner. We focus our interest now on the case $\operatorname{char}(F) = 2$. The algorithms have been shown with the previous Lie algebra. As we did with \mathfrak{e}_6 , we can construct a basis B of \mathfrak{e}_7 from the list of generators (see [4, §3.2] and Proposition 1):

$$(22) \quad S = \{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_7, \phi_1^t, \phi_2^t, \phi_3^t, \phi_4^t, \phi_5^t, \phi_6^t, \phi_7^t\},$$

where

$$\begin{aligned}
 \phi_1 &= E_{7,8} + E_{9,10} + E_{11,12} + E_{13,15} + E_{16,18} + E_{19,22} + E_{35,38} \\
 &\quad + E_{39,41} + E_{42,44} + E_{45,46} + E_{47,48} + E_{49,50}, \\
 \phi_2 &= E_{5,6} + E_{7,9} + E_{8,10} + E_{20,23} + E_{24,26} + E_{27,29} + E_{28,30} \\
 &\quad + E_{31,33} + E_{34,37} + E_{47,49} + E_{48,50} + E_{51,52}, \\
 \phi_3 &= E_{5,7} + E_{6,9} + E_{12,14} + E_{15,17} + E_{18,21} + E_{22,25} + E_{32,35} \\
 &\quad + E_{36,39} + E_{40,42} + E_{43,45} + E_{48,51} + E_{50,52}, \\
 \phi_4 &= E_{4,5} + E_{9,11} + E_{10,12} + E_{17,20} + E_{21,24} + E_{25,28} + E_{29,32} \\
 &\quad + E_{33,36} + E_{37,40} + E_{45,47} + E_{46,48} + E_{52,53}, \\
 \phi_5 &= E_{3,4} + E_{11,13} + E_{12,15} + E_{14,17} + E_{24,23} + E_{26,29} + E_{28,31} \\
 &\quad + E_{30,33} + E_{40,43} + E_{42,45} + E_{44,46} + E_{53,54}, \\
 \phi_6 &= E_{2,3} + E_{13,16} + E_{15,18} + E_{17,21} + E_{20,24} + E_{23,26} + E_{31,34} \\
 &\quad + E_{33,37} + E_{36,40} + E_{39,42} + E_{41,44} + E_{54,55}, \\
 \phi_7 &= E_{1,2} + E_{16,19} + E_{18,22} + E_{21,25} + E_{24,28} + E_{26,30} + E_{27,31} \\
 &\quad + E_{29,33} + E_{32,36} + E_{35,39} + E_{38,41} + E_{55,56},
 \end{aligned}$$

and $E_{i,j}$ the elementary matrices of order 56. We can find now a basis $B = \{b_i\}$ (<http://www.matap.uma.es/~alberca/Be7char2.txt>) with 133 elements. We compute a Cartan decomposition and obtain that

$$(23) \quad \mathfrak{e}_7 = H \oplus \left(\bigoplus_{i=1}^{63} V_i \right), \quad |H| = 6, \quad |V_i| = 2, \quad i = 1, \dots, 63.$$

with

$$(24) \quad H = \{b_{18}, b_{22}, b_{23}, b_{24}, b_{29}, b_{30}, b_{33}\},$$

a Cartan subalgebra. In this case, the center of the Lie algebra is

$$(25) \quad Z(\mathfrak{e}_7) = \langle b_{18} + b_{23} + b_{29} \rangle, \quad \text{char}(F) = 2.$$

Now, we compute the matrix of derivation d in V . As $d(H) \subset Z$, and the root spaces are d -invariant, the matrix of such a derivation is a 133×133 block diagonal, in the new basis, as

$$(26) \quad M = \text{diag}(X, X_1, \dots, X_{36}), \quad X_i = \begin{pmatrix} x_{1,i} & x_{2,i} \\ x_{3,1} & x_{4,i} \end{pmatrix},$$

where the i -row of X is as $(\beta_i \quad 0 \quad \beta_i \quad 0 \quad \beta_i \quad 0 \quad 0)$, $i = 1, \dots, 7$. The dimension of V turns out to be 7. Then

$$(27) \quad \begin{aligned} \dim \text{Der}(\mathfrak{e}_7) &= \dim \mathfrak{e}_7 + \dim V - \dim H = \\ &= 133 + 7 - 7 = 133. \end{aligned}$$

Thus, as $\dim \text{ad}(\mathfrak{e}_7) = \dim \mathfrak{e}_7 - \dim Z(\mathfrak{e}_7) = 132$, every derivation is not inner in this case, with $\text{char}(F) = 2$.

We have repeated this straightforward calculation with $\text{char}(F) = 3$. With the corresponding new basis, <http://www.matap.uma.es/~alberca/Be7char3.txt>, we obtained the Cartan decomposition:

$$(28) \quad \mathfrak{e}_7 = H \oplus \left(\bigoplus_{i=1}^{126} V_i \right), \quad |H| = 6, \quad |V_i| = 1, \quad i = 1, \dots, 126.$$

with

$$(29) \quad H = \{b_{18}, b_{21}, b_{24}, b_{26}, b_{27}, b_{28}, b_{32}\},$$

a Cartan subalgebra. The center now is

$$(30) \quad Z(\mathfrak{e}_7) = 0, \text{ char}(F) = 3,$$

and $\dim V = 7$, but now the matrix of a derivation d in V is as

$$(31) \quad M = \text{diag}(0_{7 \times 7}, X_1, \dots, X_{36}), \quad X_i = \begin{pmatrix} x_{1,i} & x_{2,i} \\ x_{3,1} & x_{4,i} \end{pmatrix},$$

as d vanishes in H . As $\dim \text{Der}(\mathfrak{e}_7) = \dim \mathfrak{e}_7 + \dim V - \dim H = 133$ and $\dim \text{ad}(\mathfrak{e}_7) = \dim \mathfrak{e}_7 - \dim Z(\mathfrak{e}_7) = 133 - 0 = 133$, every derivation is inner.

Thus we have:

\mathfrak{L}	char.	$\dim \mathfrak{L}$	$\dim Z(\mathfrak{L})$	$\dim \text{ad}(\mathfrak{L})$	$\dim \text{Der}(\mathfrak{L})$	$\text{ad}(\mathfrak{L}) = \text{Der}(\mathfrak{L})$
\mathfrak{e}_7	2	133	1	132	133	no
\mathfrak{e}_7	3	133	0	133	133	yes

and then

Theorem 7. *Let $\mathfrak{L} = \mathfrak{e}_7$ over a field F of prime characteristic p . Every derivation of \mathfrak{e}_7 is inner if, and only if, $p \neq 2$.*

4.3. Lie algebra \mathfrak{e}_8 . We face now with the most complicated case, \mathfrak{e}_8 , which is a 248-dimensional Lie algebra. In this section is where we can spread the strategy and prove the following theorem:

Theorem 8. *Every derivation of \mathfrak{e}_8 , over a field F of prime characteristic p , is inner.*

Proof. As shown in (6) we have that the killing form has determinant $2^\alpha 3^\beta 5^\gamma$, for some $\alpha, \beta, \gamma \in F$. If $\text{char}(F) \neq 2, 3, 5$, every derivation is inner by Corollary 1.

We consider now the characteristic 2 case. We have obtained a list S of generators of the Lie algebra from GAP (Picasso). We obtain then a basis applying again Proposition 1 to the set S . The basis is denoted by $B = \{b_i\}_{i=1}^{248}$ of \mathfrak{e}_8 and it is available in <http://www.matap.uma.es/~alberca/Be8char2.txt>. We will not write here the explicit expression of each b_i in terms of the elementary matrices E_{ij} .

As we did in the previous Lie algebras, we have a Cartan decomposition

$$(32) \quad \mathfrak{e}_8 = H \oplus \left(\bigoplus_{i=1}^{240} V_i \right), \quad |H| = 8, \quad |V_i| = 1, \quad i = 1, \dots, 240,$$

with 1-dimensional root spaces and

$$(33) \quad H = \{b_{18}, b_{21}, b_{23}, b_{25}, b_{29}, b_{34}, b_{37}, b_{38}\},$$

a Cartan subalgebra. Finally, we compute the center of the Lie algebra, and we get no variables left of a generic element with the condition to be in the center, and then $\dim Z(\mathfrak{e}_8) = 0$.

We have all the dimensions computed and we can affirm that

$$(34) \quad \begin{aligned} \dim \text{Der}(\mathfrak{e}_8) &= \dim \mathfrak{e}_8 + \dim \dim V - \dim H = \\ &= 248 + 8 - 8 = 248. \end{aligned}$$

Thus, as $\dim \text{ad}(\mathfrak{e}_8) = 248$, because $Z(\mathfrak{e}_8) = 0$, every derivation is inner in this case, with $\text{char}(F) = 2$.

We have repeated all this computation if $\text{char}(F) = 3$ and if $\text{char}(F) = 5$. We had to recalculate a basis from the generators but after that we only had to

change the characteristic of the base field and execute all the code. The corresponding basis are available in <http://www.matap.uma.es/~alberca/Be8char3.txt> and <http://www.matap.uma.es/~alberca/Be8char5.txt>.

We had the following Cartan decompositions, if the characteristic is 3:

$$(35) \quad \mathfrak{e}_8 = H \oplus \left(\bigoplus_{i=1}^{240} V_i \right), \quad |H| = 8, \quad |V_i| = 1, \quad i = 1, \dots, 240,$$

with 1-dimensional root spaces and

$$(36) \quad H = \{b_{19}, b_{23}, b_{24}, b_{25}, b_{26}, b_{30}, b_{33}, b_{34}\},$$

a Cartan subalgebra. In this case then the matrix M of a derivation in V is diagonal and we obtain again that $\dim V = 8$. We also confirmed that the center is also zero; and if the characteristic is 5:

$$(37) \quad \mathfrak{e}_8 = H \oplus \left(\bigoplus_{i=1}^{240} V_i \right), \quad |H| = 8, \quad |V_i| = 1, \quad i = 1, \dots, 240,$$

with 1-dimensional root spaces and

$$(38) \quad H = \{b_{23}, b_{24}, b_{26}, b_{27}, b_{28}, b_{33}, b_{37}, b_{38}\},$$

a Cartan subalgebra. In this case then the matrix M of a derivation in V is diagonal and we obtain again that $\dim V = 8$. We also confirmed that the center is also zero.

We have all the results at the following table:

\mathfrak{L}	char.	$\dim \mathfrak{L}$	$\dim Z(\mathfrak{L})$	$\dim \text{ad}(\mathfrak{L})$	$\dim \text{Der}(\mathfrak{L})$	$\text{ad}(\mathfrak{L}) = \text{Der}(\mathfrak{L})$
\mathfrak{e}_8	2	248	0	248	248	yes
\mathfrak{e}_8	3	248	0	248	248	yes
\mathfrak{e}_8	5	248	0	248	248	yes

5. FURTHER CONSIDERATIONS

We summarize all the results and the software involved at the following table:

\mathfrak{L}	char.	\dim	$Z(\mathfrak{L})$	$\text{ad}(\mathfrak{L})$	$\text{Der}(\mathfrak{L})$	inner	Software
\mathfrak{g}_2	2	14	0	14	21	no	GAP
\mathfrak{g}_2	3	14	0	14	14	yes	GAP
\mathfrak{f}_4	2	52	0	52	52	yes	GAP
\mathfrak{f}_4	3	52	0	52	52	yes	GAP
\mathfrak{e}_6	2	78	0	78	78	yes	GAP
\mathfrak{e}_6	3	78	1	77	78	no	GAP
\mathfrak{e}_7	2	133	1	132	133	no	Mathematica
\mathfrak{e}_7	3	133	0	133	133	yes	Mathematica
\mathfrak{e}_8	2	248	0	248	248	yes	GAP (Picasso) and Mathematica
\mathfrak{e}_8	3	248	0	248	248	yes	GAP (Picasso) and Mathematica
\mathfrak{e}_8	5	248	0	248	248	yes	GAP (Picasso) and Mathematica

TABLE 1. Dimensions, results and software.

And as a corollary

Theorem 9. *The derivations of Chevalley algebras of exceptional type are all inner except in the cases \mathfrak{e}_6 in characteristic 3 and \mathfrak{e}_7 in characteristic 2.*

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P. ALBERCA BJERREGAARD: DEPARTAMENTO DE MATEMÁTICA APLICADA, ESCUELA TÉCNICA SUPERIOR DE INGENIEROS INDUSTRIALES, UNIVERSIDAD DE MÁLAGA. 29071 MÁLAGA. SPAIN.
E-mail address: pgalberca@uma.es

D. MARTÍN BARQUERO: DEPARTAMENTO DE MATEMÁTICA APLICADA, ESCUELA TÉCNICA SUPERIOR DE INGENIEROS INDUSTRIALES, UNIVERSIDAD DE MÁLAGA. 29071 MÁLAGA. SPAIN.
E-mail address: dmartin@uma.es

C. MARTÍN GONZÁLEZ: DEPARTAMENTO DE ÁLGEBRA GEOMETRÍA Y TOPOLOGÍA, FACULTAD DE CIENCIAS, UNIVERSIDAD DE MÁLAGA, CAMPUS DE TEATINOS S/N. 29071 MÁLAGA. SPAIN.
E-mail address: candido@apncs.cie.uma.es